Another use of Complex Analysis in Calculus on the real line.

<u>partial fractions:</u>

$$\frac{x^2-3x+1}{(x-1)(x)(x+2)(x-3)} = \frac{A}{x-1}+\frac{B}{x}+\frac{C}{x+2}+\frac{D}{x-3}$$

using residues

$$= \frac{\frac{1-3+1}{1\cdot 3\cdot(-2)}}{x-1} + \frac{\frac{1}{(-1)(2)(-3)}}{x} + \frac{\frac{4+6+1}{(-3)(-2)(-5)}}{x+2} + \frac{\frac{9-9+1}{2\cdot 3\cdot 5}}{x-3}$$

If you have $(x-w)^2$ or $(x-z_0)^3$,

$(x=-3)$ $\quad$ $\frac{x^2+x+1}{x+3} \leftarrow x=0$ $\quad$ $\left(\frac{x^2+x+1}{x+3}\right)'$ $\quad$ $x=0$

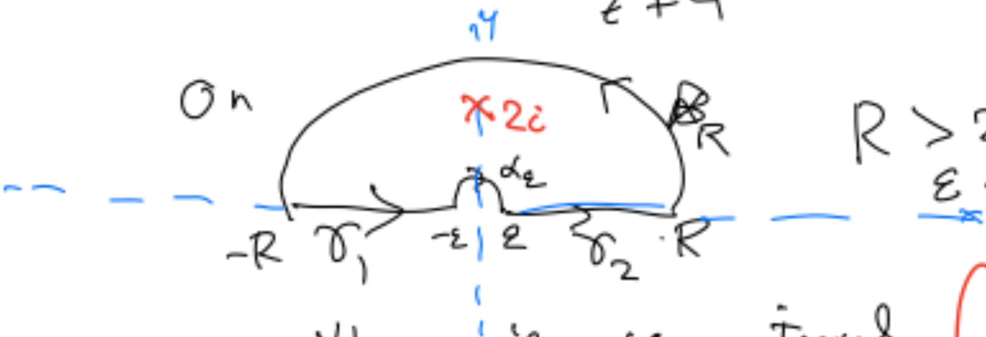
$$\frac{x^2+x+1}{x^2(x+3)} = \frac{\frac{9-3+1}{9}}{x+3} + \frac{1/3}{x^2} + \frac{\quad}{x}$$

Laurent $\frac{1}{z^2}f(z) = \frac{1}{z^2}\left(\overset{f(0)}{c_0} + \overset{f'(0)}{c_1} z + \dots\right)$

Example $\displaystyle\int_0^\infty \frac{\ln x}{x^2+4}\,dx = ?$

Consider $f(z) = \dfrac{\log(z)}{z^2+4}$ ← branch

On



$R > 2$

$\varepsilon < 2$

we will consider this integral $\displaystyle\int_{\gamma_1 \cup \alpha_\varepsilon \cup \gamma_2 \cup \beta_R} \frac{\log(z)}{z^2+4}\,dz$

$C_{R,\varepsilon}$

as $R \to \infty$, $\varepsilon \to 0^+$

$$\log(z) = \log|z| + i\arg(z)$$

$-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$.

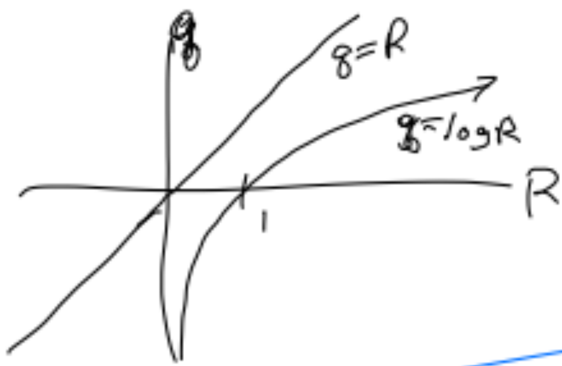
On β_R, $z = Re^{i\theta}$, $0 \le \theta \le \pi$

$$\left|\frac{\log(z)}{z^2+4}\right| = \frac{|\log(z)|}{|z^2+4|} \le \frac{|\log|z| + i\theta|}{|z|^2-4}$$

$$\le \frac{|\log R| + |\theta|}{R^2-4} = \frac{\log R + \pi}{R^2-4}$$

$$\therefore \left|\int_{\beta_R} \frac{\log(z)}{z^2+4}\,dz\right| \le \pi R \cdot \left(\frac{\log R + \pi}{R^2-4}\right) \to 0 \text{ as } R \to \infty$$

$$\pi R \cdot \left(\frac{\log R + \pi}{R^2-4}\right) = \frac{\pi \log R}{R}\left(\frac{1 + \frac{\pi}{\log R}}{1 - 4/R^2}\right)$$



$$\lim_{R\to\infty} \frac{\log R}{R} \overset{l'H}{=} \lim_{R\to\infty} \frac{\frac{1}{R}}{1} = 0.$$
$\frac{\infty}{\infty}$

for $z \in \alpha_\varepsilon$, $z = \varepsilon e^{i(\pi-\theta)}$ $0 \le \theta \le \pi$

$$\left|\frac{\log(z)}{z^2+4}\right| \le \frac{|\log\varepsilon + i(\pi-\theta)|}{4-|z|^2}$$

$$\le \frac{|\log\varepsilon| + (\pi-\theta)}{4-\varepsilon^2}$$

$$\le \frac{-\log\varepsilon + \pi}{4-\varepsilon^2}$$

Then $$\left|\int_{\alpha_\varepsilon} \frac{\log(z)}{z^2+4}\,dz\right| \le \pi\varepsilon\left(\frac{-\log\varepsilon+\pi}{4-\varepsilon^2}\right)$$

$$= \frac{-\pi\varepsilon(\log\varepsilon)\left(1-\frac{\pi}{\log\varepsilon}\right)}{4-\varepsilon^2}$$ $\to -\infty$

$$\lim_{\varepsilon\to0^+} \varepsilon\log(\varepsilon) = \lim_{\varepsilon\to0^+} \frac{\log(\varepsilon)}{1/\varepsilon} \overset{l'H}{=} \lim_{\varepsilon\to0^+} \frac{\frac{1}{\varepsilon}}{-1/\varepsilon^2}$$
$\frac{-\infty}{\infty}$
$$= \lim_{\varepsilon\to0^+}(-\varepsilon) = \boxed{0}.$$

$$\int_{\gamma_2} \frac{\log(z)}{z^2+4}\,dz = \int_{\varepsilon}^{R} \frac{\log(x)}{x^2+4}\,dx \qquad \gamma_2: z = x,\ \varepsilon < x < R$$

$$I = \lim_{\substack{\varepsilon\to 0^+\\ R\to\infty}} \left(\quad " \quad \right) = \int_0^\infty \frac{\log(x)\,dx}{x^2+4}$$

$$\int_{\gamma_1} \frac{\log(z)}{z^2+4}\,dz \qquad z = x$$

$$-R \le x \le -\varepsilon$$

$$\parallel$$

$$\int_{-R}^{-\varepsilon} \frac{\log|x| + i\overset{\arg(x)}{\pi}}{x^2+4}\,dx$$

$$u = -x \quad du = -dx$$

$$= -\int_{u=R}^{\varepsilon} \frac{\log u^{+} + i\pi}{u^2+4}\,du$$

$$= \int_{\varepsilon}^{R} \frac{\log(u) + i\pi}{u^2+4}\,du$$

$$\lim_{\substack{\varepsilon\to 0^+\\ R\to\infty}} \int_{\gamma_1} \frac{\log(z)}{z^2+4}\,dz = \int_0^\infty \frac{\log(x) + i\pi}{x^2+4}\,dx$$

$$= I + i\pi \int_0^\infty \frac{1}{x^2+4}\,dx$$

$$\int_{C_{R,\varepsilon}} \to I + I + i\pi \int_0^\infty \frac{1}{x^2+4}\,dx$$

$$\therefore\ I = \frac{1}{2}\operatorname{Re}\left(\lim_{\substack{\varepsilon\to 0^+\\ R\to\infty}} \int_{C_{R,\varepsilon}} \frac{\log(z)}{z^2+4}\,dz\right)$$

$$\overset{\text{Residue Thm}}{=} \frac{1}{2}\operatorname{Re}\left(2\pi i \cdot \operatorname{Res}\left(\frac{\log(z)}{z^2+4}, 2i\right)\right)$$

$$(z+2i)(z-2i)$$

$$\frac{\log(2i)}{4i}$$

$$= \frac{1}{2}\operatorname{Re}\left(\cancel{2}\pi \cancel{i} \cdot \left(\frac{\log|2i| + i\overset{\pi/2}{\arg}(2i)}{\overset{2}{\cancel{4}}\cancel{i}}\right)\right)$$

$$I = \boxed{\frac{\pi\log(2)}{4}}.$$

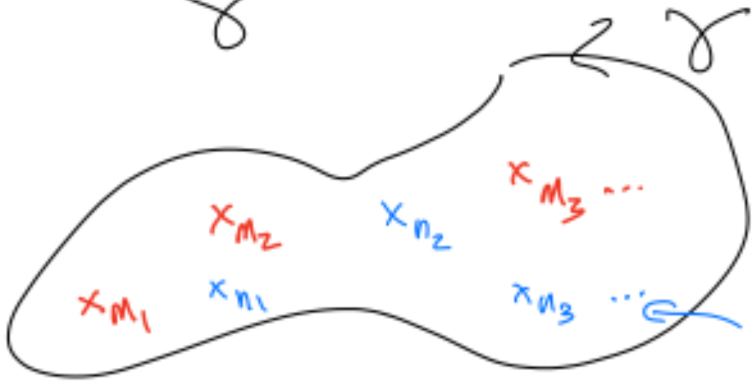
Generalize: Argument Principle: If f is meromorphic on $\text{int}(\gamma)$ and on γ (no zeros or poles on γ)

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\Big((n_1+n_2+\cdots) - (m_1+m_2+\cdots)\Big)$$

$$= 2\pi i\,(Z_f - P_f)$$

Z_f = # of zeros of f counting multiplicities

P_f = # of poles " "



poles of order m_j

zeros of order n_j

↑ very useful in counting zeros of functions.

Another interpretation

$$\frac{f'(z)}{f(z)} = \big(\log(f(z))\big)'$$

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \big(\log(f(z))\big)'\,dz$$

$$\int_a^b g'(z)\,dz = g(b)-g(a) \to \text{FTC LI}$$

FTC CI

$$= \log(f(a_-)) - \log(f(a_+))$$

$$= \cancel{\log|f(a_-)|} + i\arg(f(a_-)) - \cancel{\log|f(a_+)|} - i\arg(f(a_-))$$

$$= 2\pi i \cdot \left(\text{\# of turns clockwise that } f(z) \text{ makes around the origin}\right)$$

$f(z)=z^2$

$\gamma(t)=e^{it}$

$0\le t\le 2\pi$

image

Summary:

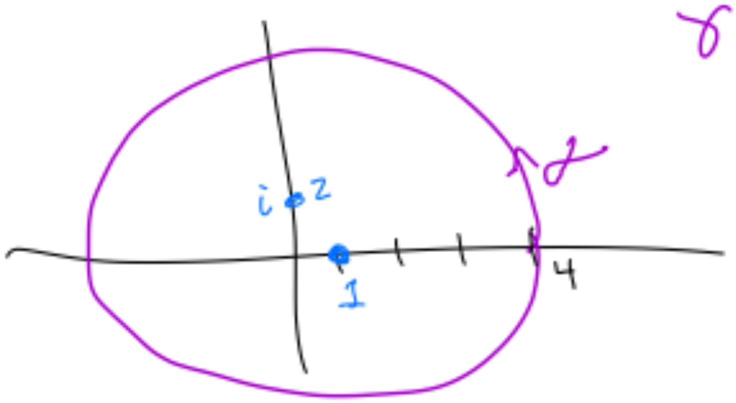
Argument Principle:

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\left(Z_f - Z_p\right)$$

$$= 2\pi i\left(\begin{array}{l}\text{\# of time } f(z) \\ \text{winds around } 0 \\ \text{when } z \text{ follows } \gamma\end{array}\right)$$

E.g. $f(z) = (z-1)(z+i)^2$

γ: circle of radius 4 ccw.



$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\,(Z_f - P_f)$$

$$= 2\pi i\,(3-0) = 6\pi i$$

$$= 2\pi i \left(\begin{array}{c} \#\text{ of times} \\ (z-1)(z+i)^2 \text{ revolves} \\ \text{around } w=0 \text{ as} \\ z \text{ follows } \gamma \end{array} \right)$$

revolves 3 times.

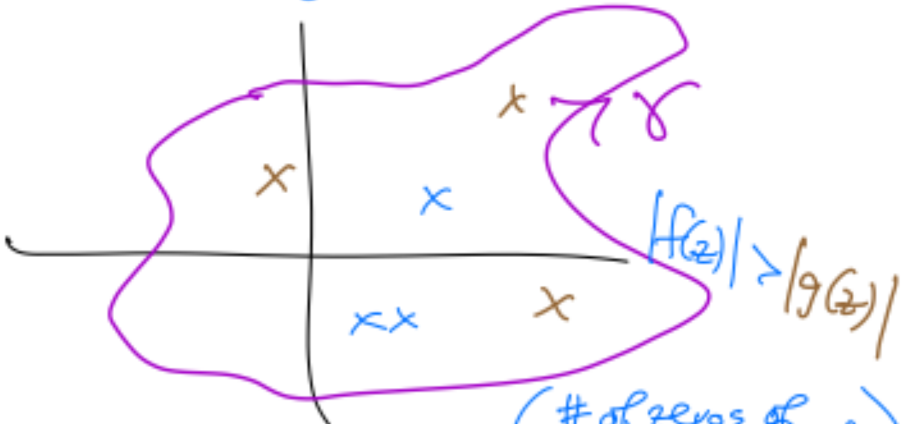
Same answer for this γ:

Corollary:

Thm (Rouché's Thm). Suppose f & g are holomorphic on a simple closed ~~rect~~ ~~piecewise~~ smooth curve γ and on the interior of γ. (Say γ is positively oriented)

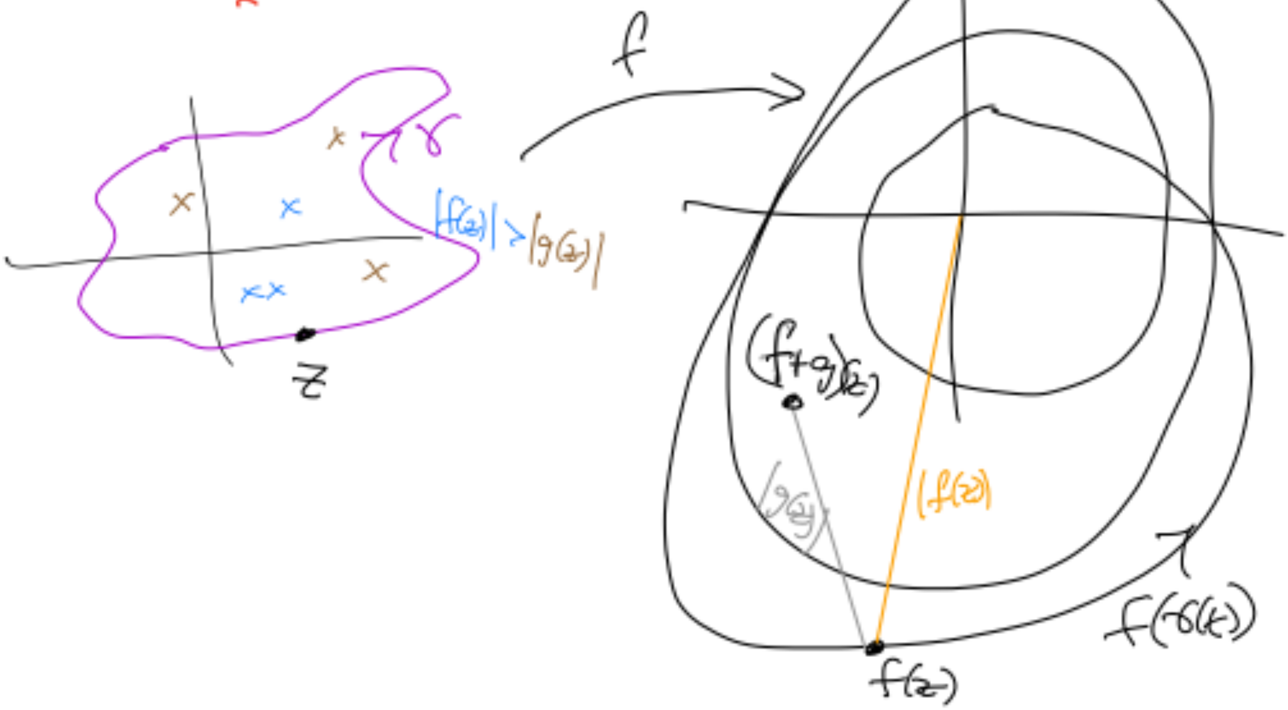
Suppose
- f & g have no zeros on γ
- $|f(z)| > |g(z)|$ for $z \in \gamma$.

Then f and $f+g$ have the same number of zeros in the interior of γ.



$\left(\begin{array}{c}\#\text{ of zeros of } f \text{ inside}\end{array}\right) = \left(\begin{array}{c}\#\text{ of zeros}\\ \text{of } f+g \text{ inside}\end{array}\right)$

Idea of proof:



$\Rightarrow (f+g)(z)$ must wrap around 0 the same # of times as $f(z)$ as z follows γ.